Journal of Computational and Applied Mathematics 233 (2009) 1175–1186

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The self-validated method for polynomial zeros of high efficiency

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ARTICLE INFO

Article history:
Received 24 May 2008
Received in revised form 9 September 2009

MSC:
65H05
65G20
30C15

Keywords:
Zeros of polynomials
Simultaneous methods
Inclusion methods
Circular interval arithmetic
Acceleration of convergence
Computational efficiency

ABSTRACT

The improved iterative method of Newton's type for the simultaneous inclusion of all simple complex zeros of a polynomial is proposed. The presented convergence analysis, which uses the concept of the R-order of convergence of mutually dependent sequences, shows that the convergence rate of the basic third order method is increased from 3 to 6 using Ostrowski's corrections. The new inclusion method with Ostrowski's corrections is more efficient compared to all existing methods belonging to the same class. To demonstrate the convergence properties of the proposed method, two numerical examples are given.

1. Introduction

Iterative interval methods for the simultaneous inclusion of polynomial complex zeros produce intervals (disks or rectangles) that contain these zeros in each iteration. For this reason, this class of methods can be regarded as a self-validated numerical tool that provides automatic computation of rigorous error bound (given by the semi-width of the resulting inclusion intervals) on approximate solutions. This very useful property is the main advantage of inclusion methods whose remarkable development started at the seventies of the 20th century with the rapid growth of digital computers. A considerable amount of the applied interval process is aimed at improving the approximate result and giving error bounds for the improved approximations.

The purpose of this paper is to continue the study presented in [1,2] concerned with the Newton-like method for the simultaneous inclusion of the zeros of algebraic polynomials. We propose a self-validated zero-finding method in circular complex arithmetic with accelerated convergence and a very high computational efficiency. The improvement is attained by applying the centered inversion instead of the exact inversion of disks and using Ostrowski's correction [3, p. 253].

The presentation of the paper is organized as follows. The basic properties of circular complex arithmetic are given in the Introduction. The Newton-like interval method and its modification are studied in Section 2. The improved inclusion method with Ostrowski's corrections and its convergence analysis are proposed and studied in Section 3, while its single step variant is considered in Section 4. The analysis of computational efficiency and numerical results are presented in Section 5.

The construction and convergence analysis of the proposed algorithms require the basic properties of the so-called circular complex arithmetic introduced in [1]. A circular closed region (disk) $Z := \{z : |z - c| \le r\}$ with center $c := \operatorname{mid} Z$ and radius $r := \operatorname{rad} Z$ will be denoted by the parametric notation $Z := \{c; r\}$. We will use the abbreviation INV to denote

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0377-0427/$ – see front matter © 2009 Published by Elsevier B.V.
doi:10.1016/j.cam.2009.09.016

the inversion of a disk. If $Z_k = \{c_k; r_k\}$ $(k = 1, 2)$, then

$$Z_1 \pm Z_2 = \{c_1 \pm c_2; r_1 + r_2\},$$
$$w \cdot Z = \{w \,\mathrm{mid}\, Z; |w| \mathrm{rad}\, Z\} \quad (w \in \mathbb{C}),$$
$$Z_1 \cdot Z_2 = \{c_1c_2; |c_1|r_2 + |c_2|r_1 + r_1r_2\},$$
$$Z^{-1} = \{c; r\}^{-1} = \left\{ \frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2} \right\} \quad (0 \notin Z) \text{ (exact inversion)}, \tag{1}$$
$$Z^{I_c} = \{c; r\}^{I_c} = \left\{ \frac{1}{c}; \frac{r}{|c|(|c| - r)} \right\} \supset Z^{-1} \quad (0 \notin Z) \text{ (centered inversion)}, \tag{2}$$
$$Z_1 : Z_2 = Z_1 \cdot \mathrm{INV}\, Z_2 \quad (0 \notin Z_2,\ \mathrm{INV} \in \{()^{-1}, ()^{I_c}\}),$$
$$z \in \{c; r\} \iff |z - c| \le r, \tag{3}$$
$$\{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset \iff |c_1 - c_2| > r_1 + r_2. \tag{4}$$

The addition, subtraction and inversion Z^{-1} are exact operations, that is, $Z_1 * Z_2 = \{z_1 * z_2 : z_1 \in Z_1, z_2 \in Z_2\}, * \in \{+, -, ()^{-1}\}$.

For the basic interval operations $+, -, \cdot, :$ the *inclusion property* is valid, that is,

$$Z_k \subseteq W_k \Longrightarrow Z_1 * Z_2 \subseteq W_1 * W_2 \quad (k = 1, 2; * \in \{+, -, \cdot, :\}).$$

An interval function F is called *complex circular extension* of a complex function f if

$$F(z) = f(z), \quad (z \in Z), \qquad F(Z) \supseteq \{f(z) : z \in Z\}.$$

If f is a rational function and F is its complex circular extension, then

$$Z_k \subseteq W_k \ (k = 1, \ldots, q) \Longrightarrow F(Z_1, \ldots, Z_q) \subseteq F(W_1, \ldots, W_q).$$

In particular, we have

$$w_k \in W_k \ (k = 1, \ldots, q;\ w_k \in \mathbb{C}) \Rightarrow f(w_1, \ldots, w_q) \in F(W_1, \ldots, W_q).$$

More details about circular arithmetic can be found in the books [4, Chapter 5] and [5, Chapter 1].

2. Newton-like interval method with corrections

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = \prod_{j=1}^{n}(z - \zeta_j)$ be a monic polynomial of degree n with simple real or complex zeros $\zeta_1, \ldots, \zeta_n$ and let

$$u(z) = \frac{P(z)}{P'(z)} = \left[\frac{\mathrm{d}}{\mathrm{d}z} \ln P(z) \right]^{-1} = \left(\sum_{j=1}^{n} \frac{1}{z - \zeta_j} \right)^{-1} \tag{5}$$

be Newton's correction appearing in the quadratically convergent Newton's method $\hat{z} = z - u(z)$. From (5) we derive the following fixed point relation

$$\zeta_i = z - \frac{1}{\frac{1}{u(z)} - \sum\limits_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{z - \zeta_j}} \quad (i \in \boldsymbol{I}_n := \{1, \ldots, n\}). \tag{6}$$

Let $z_1, \ldots, z_n$ be distinct approximations to the zeros $\zeta_1, \ldots, \zeta_n$. Putting $z = z_i$ and substituting the zeros ζ_j by their approximations z_j in (6), we obtain the iterative method of the third order

$$\hat{z}_i = z_i - \frac{1}{\frac{1}{u(z_i)} - \sum\limits_{\substack{j=1 \\ j \ne i}}^{n} \frac{1}{z_i - z_j}} \quad (i \in \boldsymbol{I}_n) \tag{7}$$

for the simultaneous determination of all simple zeros of the polynomial P. Here $\hat{z}_i$ is a new approximation to the zero ζ_i. The iterative method (7) was noticed in [6,7], but its practical application for the simultaneous computation of polynomial zeros appeared for the first time in the papers in [8,9], together with the proof of cubic convergence. For this reason, the method (7) is most frequently referred to as the Ehrlich–Aberth method.

To construct the third order method (7), the zeros ζ_j in (6) are replaced by the current approximations z_j. It is clear from (6) that the better approximation to ζ_j would give the faster method than the Ehrlich–Aberth method (7). Nourein [10] realized this idea taking Newton's approximations $z_j - u(z_j)$ (instead of z_j) to obtain the fourth order method, often called Nourein's method,

$$\hat{z}_i = z_i - \frac{1}{\frac{1}{u(z_i)} - \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{z_i - z_j + u(z_j)}} \quad (i \in \boldsymbol{I}_n). \tag{8}$$

We immediately observe that the increase of the convergence order from 3 to 4 is carried out using the already calculated corrections $u(z_j)$ and only n additional subtractions $z_j - u(z_j)$ per iteration. Therefore, the very fast method is constructed with neglect number of additional numerical operations, which points to a high computational efficiency of the Nourein method (8).

Let $Z_1^{(0)}, \ldots, Z_n^{(0)}$ be an array of disjoint disks containing isolated zeros of P, that is, $\zeta_i \in Z_i^{(0)}$ $(i \in \boldsymbol{I}_n)$. Starting from the fixed point relation (6), Gargantini and Henrici developed in [1] the following method for the simultaneous inclusion of polynomial zeros

$$Z_i^{(m+1)} = z_i^{(m)} - \frac{1}{\frac{1}{u\left(z_i^{(m)}\right)} - \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \left(z_i^{(m)} - Z_j^{(m)}\right)^{-1}} \quad (i \in \boldsymbol{I}_n;\ m = 0, 1, 2 \ldots), \tag{9}$$

realized in complex circular interval arithmetic, where $z_i^{(m)} = \operatorname{mid} Z_i^{(m)}$.

In the sequel we will often omit the iteration index m and denote the quantities in the next iteration with the symbol ˆ(hat). The iterative formula (9) can be rewritten in the form

$$\widehat{Z}_i = z_i - u(z_i) \cdot \frac{1}{1 - u(z_i) \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} (z_i - Z_j)^{-1}} \quad (i \in \boldsymbol{I}_n),$$

wherefrom we see that (9) resembles Newton's method. For this reason, in literature the interval method (9) is often called the Newton-like interval method. It was proved in [1] that the order of convergence of the interval method (9) is three, that is, the sequences of the radii $\left\{\operatorname{mid} Z_i^{(m)}\right\}_{i=1,\ldots,n}$ have the cubic convergence.

Remark 1. Some authors often claim that quadratically convergent iterative methods for solving equations are most efficient and give quite satisfactory results in practice so that there is no necessity for the construction of methods of higher order. This assertion is perhaps acceptable in the case of methods realized in ordinary complex arithmetic but it does not hold in circular interval arithmetic. Namely, the second order iterative method for the simultaneous inclusion of complex zeros of the form (see [11], [12, Chapter 3])

$$\widehat{Z}_i = z_i - \frac{P(z_i)}{\prod\limits_{\substack{j=1 \\ j\neq i}}^{n} (z_i - Z_j)} \quad (i \in \boldsymbol{I}_n;\ z_i = \operatorname{mid} Z_i)$$

is more expensive than the Newton-like interval method (9). Moreover, since the product of disks is not an exact operation and, thus, always gives an enlarged disk, it can happen that the denominator encloses the number 0 and the iterative process falls down. For this reason, the construction of higher order interval methods is quite justified.

One believed for two decades that the convergence order of the Newton-like method (9) cannot be increased without considerable increase of computational cost. However, using the already mentioned idea with corrections applied in (8), Carstensen and Petković [2] accelerated the interval method (9) requiring neglect number of additional numerical operations in this manner:

$$\widehat{Z}_i = z_i - \mathrm{INV}_2\left(1/u(z_i) - \sum_{\substack{j=1 \\ j\neq i}}^{n} \mathrm{INV}_1\left(z_i - Z_j + u(z_j)\right)\right) \quad (i \in \boldsymbol{I}_n). \tag{10}$$

Here the subscript indices "1" and "2" of INV point to the order of application of the inversions. We note that the midpoints of the disks $\operatorname{mid}(Z_j - u(z_j)) = z_j - u(z_j)$ behave as the approximations obtained by Newton's method, that eventually provides the acceleration of convergence of the sequences of radii $\left\{\operatorname{rad} Z_i^{(m)}\right\}$. The detailed analysis was given in [2].

Let $O_R(\mathrm{IM})$ denote the R-order of convergence of an iteration method IM. The following assertion was proved in [2].

Theorem 1. *If initial inclusion disks $Z_1^{(0)}, \ldots, Z_n^{(0)}$ are reasonably small, then the R-order of convergence of the interval method (10) is given by*

$$O_R(10) \geq \begin{cases} (3+\sqrt{17})/2 \cong 3.562 & \text{if } \mathrm{INV}_1 = ()^{-1}, \\ 4 & \text{if } \mathrm{INV}_1 = ()^{I_c}. \end{cases}$$

In essence, the increase of the convergence rate is the result of the accelerated convergence of the midpoints of the disks $\hat{Z}_i$ calculated by (10). Namely, the sequences $\{\operatorname{mid} \hat{Z}_i\}$ behave as the sequences of approximations defined by Nourein's method (8). Since the convergence of the midpoints and the convergence of the radii are mutually coupled, the improvement of the midpoints by (8) improves the convergence of the radii. To be more precise, let us present the expression of the midpoints and radii of disks $Z_i^{(m)}$ (see [2])

$$\operatorname{rad} Z_i^{(m+1)} = O\Big(\big(r^{(m)}\big)^3\Big) \quad \text{(method (9) without corrections)} \tag{11}$$

and

$$\operatorname{rad} Z_i^{(m+1)} = O\Big(\big|\varepsilon^{(m)}\big|^2 r^{(m)}\Big), \qquad |\varepsilon^{(m+1)}| = O\Big(\big|\varepsilon^{(m)}\big|^4\Big)$$
$$\text{(method (10) with Newton's corrections and INV} = ()^{I_c}), \tag{12}$$

where

$$|\varepsilon^{(m)}| = \max_{1\le j\le n} \left|\operatorname{mid} Z_j^{(m)} - \zeta_j\right|, \qquad r^{(m)} = \max_{1\le j\le n} \operatorname{rad} Z_j^{(m)}.$$

The convergence order of the interval method (9) is only three, see (11). On the other hand, the introduced corrections in (10) remove the centers of disks Z_j to the improved (Newton's) approximations $z_j - u(z_j)$ instead of z_j, more closer to the exact zeros (see also the fixed point relation (6)). In this way the convergence of the midpoints of disks $Z_i^{(m+1)}$ is considerably accelerated, which additionally increases the convergence of the radii, proportionally to the square of errors $|\operatorname{mid} Z_j^{(m)} - \zeta_j|$, see the expressions (12). For more details see the book [5, Chapter 6] and the papers [2,13–15]. It is convenient to compare (12) with the relation (22) for the new method (16).

Remark 2. According to Theorem 1, the convergence order of (10) is four if INV_1 is the centered inversion, independently of the type of the inversion INV2. This holds in a limit process if the number of iteration is large. In practice, better results are obtained when we take $\mathrm{INV}_2 = ()^{I_c}$ since the centers of $\hat{Z}_i$ (produced by (10)) entirely coincide with the iterative formula (8) (of the cubic convergence), while $\mathrm{INV}_2 = ()^{-1}$ gives only some approximations of (8).

Remark 3. At first sight, faster convergence of the interval method (10) with the centered inversion (2) seems paradoxical having in mind that the centered inversion produces larger disks than the exact inversion (1). The explanation lies in the fact that the application of centered inversion provides better convergence of the midpoints of disks produced by (10), see Remark 2.

It is worth noting that the increase of convergence of the method (10) is achieved with neglect number of addition calculations; namely, the already calculated Newton's approximations $u(z_j)$ are reused, compare (9) and (10). It is assumed that the improved disks $Z_j - u(\operatorname{mid} Z_j)$ are calculated in advance to avoid repeat calculations under the sum. In this way, the increased convergence is obtained "at the price" of only n additional subtractions $Z_j - u(\operatorname{mid} Z_j)$ $(j = 1, \ldots, n)$ per iteration. As result, the computational efficiency of the interval method (10) is increased in reference to the method (9).

3. Newton-like method with Ostrowski's corrections

In this section we will present further improvement of the Newton-like interval methods. According to the previous discussion, we observe that, in general, it is desirable to accelerate the convergence of centers of disks appearing in iterative formulas of the type (9) or (10). Keeping in sight the relation (6) we see that these centers approach the zeros ζ_j, moving the center of the improved disk $\hat{Z}_i$ very close to ζ_i. Therefore, continuing to improve the convergence rate of the interval methods (9) and (10), it is preferable to apply more rapid method instead of Newton's method, but on the account of as small as possible computational cost. The following iterative method for solving nonlinear equations $f(z) = 0$, proposed in [3], is convenient in the realization of the mentioned goal:

$$\hat{z} = \phi(z) := z - u(z)\frac{f(z - u(z)) - f(z)}{2f(z - u(z)) - f(z)}, \qquad u(z) = \frac{f(z)}{f'(z)}. \tag{13}$$

This is, actually, two step method; first we calculate Newton's correction $u(z)$ and then calculate the improved approximation $\hat{z}$ by (13). The order of convergence of the Ostrowski method (13) is four; if ζ is a simple zero of f and $\varepsilon = z - \zeta$, then

$$\frac{\phi(z) - \zeta}{(z - \zeta)^4} \to A_2(\zeta)\big[A_2^2(\zeta) - A_3(\zeta)\big], \qquad A_k(z) = \frac{f^{(k)}(z)}{k!f'(z)},$$

or in the form

$$\phi(z) - \zeta = \mathcal{O}_M(\varepsilon^4), \tag{14}$$

where $\mathcal{O}_M$ is a symbol which points to the fact that two complex numbers w_1 and w_2 have magnitudes of the same order (that is, $|w_1| = \mathcal{O}(|w_2|)$, written as $w_1 = \mathcal{O}_M(w_2)$).

Let $f \equiv P$ be the monic polynomial, then (13) can be rewritten in the form

$$\hat{z}_i = \phi(z_i) = z_i - \psi(z_i), \qquad \psi(z) = u(z)\frac{P(z-u(z)) - P(z)}{2P(z-u(z)) - P(z)}, \qquad u(z) = \frac{P(z)}{P'(z)}.$$

The function $z \mapsto \psi(z)$ will be called Ostrowski's correction. Now we can derived the Ehrlich–Aberth method with Ostrowski's corrections following the idea used in the construction of the method (8):

$$\hat{z}_i = z_i - \frac{1}{\frac{1}{u(z_i)} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{z_i - z_j + \psi(z_j)}} \quad (i \in \boldsymbol{I}_n). \tag{15}$$

To our knowledge, this is a new iterative formula not derived previously in literature. We can expect very fast convergence of the method (15) since Ostrowski's approximation $z_j - \psi(z_j)$ is very close to the exact zero ζ_j (compare with (6)) due to the fact that Ostrowski's method (13) is of the fourth order. The convergence speed of the iterative method (15) is given in Theorem 4.

In a similar way we construct the Newton-like method with Ostrowski's corrections $\psi(z_j)$ in circular complex arithmetic:

$$\widehat{Z}_i = z_i - \mathrm{INV}_2\left(1/u(z_i) - \sum_{\substack{j=1\\ j\neq i}}^{n} \mathrm{INV}_1\Big(z_i - Z_j + \psi(z_j)\Big)\right) \quad (i \in \boldsymbol{I}_n). \tag{16}$$

We will determine now the R-order of convergence of the improved interval method (16). Following Remark 3 we will assume that only the centered inversion (2) is applied, that is, $\mathrm{INV}_1 = \mathrm{INV}_2 = ()^{I_c}$.

Let IM be an iterative numerical method which generates k sequences $\{z_1^{(m)}\}, \ldots, \{z_k^{(m)}\}$ for the approximations to the solutions $z_1^*, \ldots, z_k^*$. In order to estimate the order of convergence of the iterative method IM we introduce the error sequences

$$h_i^{(m)} = |z_i^{(m)} - z_i^*| \quad (i = 1, \ldots, k).$$

The order of convergence of inclusion methods with corrections can be suitably determined using the following assertion (see Theorem 3 in [16]):

Theorem 2. *Given the error recursion*

$$h_i^{(m+1)} \le \alpha_i \prod_{j=1}^{k} \big(h_j^{(m)}\big)^{t_{ij}}, \quad (i = 1, \ldots, k;\ m \ge 0), \tag{17}$$

where $t_{ij} \ge 0$, $\alpha_i > 0$, $1 \le i, j \le k$. Denote the matrix of exponents appearing in (17) with T_k, that is $T_k = [t_{ij}]_{k\times k}$. If the non-negative matrix T_k has the spectral radius $\rho(T_k) > 1$ and a corresponding eigenvector $\boldsymbol{x}_\rho > 0$, then all sequences $\{h_i^{(m)}\}$ $(i = 1, \ldots, k)$ have the R-order equals at least $\rho(T_k)$.

The matrix $T_k = [t_{ij}]$, concerned with the R-order of convergence, is usually called the *R-matrix*. Using Theorem 2 we can state the following convergence theorem.

Theorem 3. *Let $(Z_1^{(0)}, \ldots, Z_n^{(0)}) := (Z_1, \ldots, Z_n)$ be an array of disjoint initial disks containing the zeros $\zeta_1, \ldots, \zeta_n$ of P. If the midpoints of initial disks are close enough to the zeros of P, then the R-order of convergence of the iterative method (16) with the centered inversions is at least six.*

Proof. For brevity, let $\varepsilon_i = z_i - \zeta_i$, $\hat{\varepsilon}_i = \hat{z}_i - \zeta_i$ and

$$c_{ij} = z_i - z_j + \psi(z_j), \qquad \rho_i = \sum_{j\neq i} \frac{r_j|\varepsilon_i|}{|c_{ij}|(|c_{ij}| - r_j)}, \qquad \theta_i = \sum_{j\neq i} \frac{z_j - \psi(z_j) - \zeta_j}{(z_i - \zeta_j)c_{ij}}.$$

Then, starting from (16) and using (5) and circular arithmetic operations, we obtain

$$\begin{aligned}
\hat{Z}_i &= z_i - \frac{1}{\frac{1}{\varepsilon_i} + \sum\limits_{j\neq i} \frac{1}{z_i - \zeta_j} - \sum\limits_{j\neq i} \left\{ \frac{1}{c_{ij}};\ \frac{r_j}{|c_{ij}|(|c_{ij}| - r_j)} \right\}} \\
&= z_i - \frac{\varepsilon_i}{\left\{ 1 - \varepsilon_i \sum\limits_{j\neq i} \frac{z_j - \psi(z_j) - \zeta_j}{(z_i - \zeta_j)c_{ij}};\ \sum\limits_{j\neq i} \frac{r_j|\varepsilon_i|}{|c_{ij}|(|c_{ij}| - r_j)} \right\}} \\
&= z_i - \frac{\varepsilon_i}{\{1 - \varepsilon_i\theta_i;\ \rho_i\}},
\end{aligned}$$

hence

$$\hat{Z}_i = \left\{ z_i - \frac{\varepsilon_i}{1-\varepsilon_i\theta_i}; \frac{|\varepsilon_i|\rho_i}{|1-\varepsilon_i\theta_i|(|1-\varepsilon_i\theta_i|-\rho_i)} \right\}. \tag{18}$$

From (18) we find

$$\hat{z}_i = \operatorname{mid}\hat{Z}_i = z_i - \frac{\varepsilon_i}{1-\varepsilon_i\theta_i},$$

so that

$$\hat{\varepsilon}_i = \hat{z}_i - \zeta_i = \varepsilon_i - \frac{\varepsilon_i}{1-\varepsilon_i\theta_i} = \frac{-\varepsilon_i^2\theta_i}{1-\varepsilon_i\theta_i}.$$

Let us assume that $\varepsilon_i = \mathcal{O}_M(\varepsilon_j)$ for any pair i, j and let $\varepsilon \in \{\varepsilon_1, \dots, \varepsilon_n\}$ be the error of maximal modulus. Then, according to (14), we have $\theta_i = \mathcal{O}_M(\varepsilon^4)$ and from the last relation we find

$$\hat{\varepsilon}_i = \mathcal{O}_M(\varepsilon^6). \tag{19}$$

Therefore, the centers of disks $\hat{Z}_i$ obtained by the interval method (16) converge with the order 6. From (19) we obtain

$$|\hat{\varepsilon}| = \mathcal{O}(|\varepsilon|^6). \tag{20}$$

Starting from (18) we find

$$\hat{r}_i = \operatorname{rad}\hat{Z}_i = \frac{|\varepsilon_i|\rho_i}{|1-\varepsilon_i\theta_i|(|1-\varepsilon_i\theta_i|-\rho_i)}. \tag{21}$$

Since $\rho_i = \mathcal{O}(|\varepsilon_i|r_i)$ and $\theta_i = \mathcal{O}_M(\varepsilon^4)$, the denominator of (21) tends to 1 so that we have the following estimation

$$\hat{r} = \mathcal{O}(|\varepsilon|^2 r), \qquad r = \max_{1\le i\le n} r_i. \tag{22}$$

Let us introduce the abbreviations

$$r_i^{(m)} = \operatorname{mid} Z_i^{(m)}, \qquad r^{(m)} = \max_{1\le i\le n} r_i^{(m)}, \qquad |\varepsilon^{(m)}| = \max_{1\le i\le n} |\varepsilon_i^{(m)}|,$$

where $m = 0, 1, \dots$ is the iteration index. Having in mind (20) and (22), by induction we can derive the following relations

$$|\varepsilon^{(m+1)}| = \mathcal{O}\left(|\varepsilon^{(m)}|^6\right), \qquad r^{(m)} = \mathcal{O}\left(|\varepsilon^{(m)}|^2 r^{(m)}\right) \tag{23}$$

of the form (17).

The sequences $\{z_i^{(m)}\}$ and $\{r_i^{(m)}\}$ of the centers and the radii of the disks $Z_i^{(m)}$ obtained by the interval method (16) are mutually dependent so that we use Theorem 2. For simplicity, as is usual in this type of analysis, we adopt $1 > |\varepsilon^{(0)}| = r^{(0)} > 0$ which means that we deal with the "worst case" model. This assumption has no influence on the final result of the limit process which we apply in order to obtain the lower bound of the R-order of convergence.

From the relations (23) we form the R-matrix $T_2 = \begin{bmatrix} 6 & 0 \\ 2 & 1 \end{bmatrix}$ with the spectral radius $\rho(T_2) = 6$ and the corresponding eigenvector $\boldsymbol{x}_\rho = (5, 2) > 0$. Hence, according to Theorem 2, we obtain

$$O_R((16)) \le \rho(T_2) = 6,$$

which completes the proof of Theorem 3. □

Taking into account the relation (20) and the fact that the approximations $\hat{z}_i$ defined by (15) coincide with the midpoints $\operatorname{mid}\hat{Z}_i$ of the disks generated by the inclusion method (16), we immediately have the following assertion:

Theorem 4. *If initial approximations $z_1^{(0)}, \dots, z_n^{(0)}$ are sufficiently close to the zeros $\zeta_1, \dots, \zeta_n$ of a given polynomial P, then the order of convergence of the simultaneous method (15) is six.*

From (13) and (15) we observe that only n additional computations of the polynomial P (at the points $z_1 - u(z_1), \dots, z_n - u(z_n)$) provide the increase of the convergence order from 3 (of the method (7)) to 6 (of the method (15)). Hence, we can conclude that the method (15) is one of the most efficient methods for the simultaneous determination of polynomial zeros.

According to the theoretical order of convergence given in Theorems 1 and 3 it follows that the algorithms (10) and (16) with corrections are highly efficient but only if Newton's corrections $u(z_j)$ and Ostrowski's corrections $\psi(z_j)$ are sufficiently small in magnitude. Otherwise, the corrections $u(z_j)$ and $\psi(z_j)$ do not accelerate the convergence of midpoints of the produced inclusion disks and, consequently, the convergence rate of the applied interval methods (10) and (16) is smaller in practice. Moreover, it can happen that the implications

$$\zeta_i \in Z_j \Longrightarrow \zeta_i \in Z_j - u(z_j) \quad \text{(for the method (10))}$$

and

$$\zeta_i \in Z_j \Longrightarrow \zeta_i \in Z_j - \psi(z_j) \quad \text{(for the method (16))} \tag{24}$$

do not valid, which leads to the loss of inclusion property; in other words, some zeros can drop out of the calculated disks.

Now we will give an analysis concerned with the validity of the implication (24). Conditions related to the interval method (10) were studied in [2].

Let us define the measure of the separation of inclusion disks from each other by

$$\eta_i^{(m)} = \min_{\substack{1\le i,j\le n\\ j\ne i}} \left\{|z_i^{(m)} - z_j^{(m)}| - r_j^{(m)}\right\}.$$

The convergence behavior of interval simultaneous methods in circular complex arithmetic depends on many factors, but the size of initial disks and their separation are the most influential. It turned out that these features can be suitably involved in the inequality

$$\eta^{(0)} > c_n r^{(0)},$$

where c_n is a constant depending only on the polynomial degree n, see the book [12] and the references cited therein.

Let

$$\omega(z_i) = \omega_i = \frac{P(z_i - u_i)}{2P(z_i - u_i) - P(z_i)},$$

where we put $u_i = u(z_i)$ and $\omega_i = \omega(z_i)$ for simplicity. Then Ostrowski's correction can be expressed as

$$\psi(z_i) = u_i(1 - \omega_i).$$

Remark 4. Using the Taylor series we develop ω_i and obtain

$$\omega_i = \frac{u_i\big(-3P''(z_i) + P'''(z_i)u_i\big)}{6P'(z_i) + 2u_i\big(-3P''(z_i) + P'''(z_i)u_i\big)} + \mathcal{O}_M(u_i^3).$$

Hence we conclude that if z_i is a good approximation to the zero ζ_i, then $|u_i|$ is a small quantity and hence, $|\omega_i|$ is also sufficiently small quantity.

Conditions for the validity of the implication (24) are considered in the following lemma.

Lemma 1. *Let $Z_1, \ldots, Z_n$ be inclusion disks of the zeros $\zeta_1, \ldots, \zeta_n$, $\zeta_i \in Z_i$. If the inclusion disks $Z_1, \ldots, Z_n$ are chosen so that $\max_{1\le i\le n} |\omega_i| < 1/3$ and the inequality*

$$\eta > 3(n-1)r \tag{25}$$

holds, then the following implication

$$\zeta_i \in Z_i \Longrightarrow \zeta_i \in Z_i - \psi(z_i) \quad (z_i = \operatorname{mid} Z_i)$$

is valid for every $i \in \boldsymbol{I}_n$.

Proof. With regard to (3) we should prove the implication

$$|z_i - \zeta_i| = |\varepsilon_i| \le r_i \Longrightarrow |z_i - \psi(z_i) - \zeta_i| \le r_i.$$

Let $\sigma_i = \sum_{j\ne i}(z_i - \zeta_j)^{-1}$. Starting from the triangle inequality

$$|z_i - \zeta_j| \ge |z_i - z_j| - |z_j - \zeta_j| \ge |z_i - z_j| - r_j \ge \eta$$

and (25), we get

$$|\varepsilon_i\sigma_i| \le r_i \sum_{j\ne i} |z_i - \zeta_j|^{-1} \le \frac{(n-1)r}{\eta} < \frac{1}{3}. \tag{26}$$

Since

$$u_i = \frac{P(z_i)}{P'(z_i)} = \frac{1}{\sum_{j=1}^{n}(z_i - \zeta_j)^{-1}} = \frac{1}{1/\varepsilon_i + \sigma_i} = \frac{\varepsilon_i}{1 + \varepsilon_i\sigma_i},$$

taking into account that $|\varepsilon_i| \le r_i$ and (26) we estimate

$$\begin{aligned}|z_i - \psi(z_i) - \zeta_i| &= |\varepsilon_i - u_i(1-\omega_i)| = \left|\varepsilon_i - \frac{\varepsilon_i}{1+\varepsilon_i\sigma_i}(1-\omega_i)\right| \\ &= |\varepsilon_i|\left|\frac{\varepsilon_i\sigma_i + \omega_i}{1+\varepsilon_i\sigma_i}\right| \le |\varepsilon_i|\left(\frac{|\varepsilon_i\sigma_i|}{1-|\varepsilon_i\sigma_i|} + \frac{|\omega_i|}{1-|\varepsilon_i\sigma_i|}\right) \\ &< |\varepsilon_i|\left(\frac{1/3}{1-1/3} + \frac{1/3}{1-1/3}\right) \le r_i. \quad \square\end{aligned}$$

According to Remark 4 we observe that the requirement $|\omega_i| < 1/3$ of Lemma 1 can be easily fulfilled if the midpoints of the disks $Z_1, \ldots, Z_n$ are reasonable close to the zeros $\zeta_1, \ldots, \zeta_n$.

To preserve great efficiency and inclusion property of interval methods with corrections, it is preferable sometimes to apply a combined method consists of two steps:

(i) At the beginning of iterative procedure, apply one or at most two iterations of the Newton-like method (9) (without corrections);
(ii) Implement the interval method (10) or (16) with corrections.

Let us introduce the switch function

$$s_k(m) = \begin{cases} 0 & \text{if } m \le k, \\ 1 & \text{if } m > k, \end{cases}$$

where m is the iteration index and k (usually ≤ 2) is the number of iterations when the Newton-like method (9) is running. Then we modify the interval methods (10) and (16) to provide an automatic procedure in the following way:

$$Z_i^{(m+1)} = z_i^{(m)} - \mathrm{INV}_2\left(1/u(z_i^{(m)}) - \sum_{\substack{j=1\\ j\ne i}}^{n} \mathrm{INV}_1\left(z_i^{(m)} - Z_j^{(m)} + u(z_j^{(m)}) \cdot s_k(m)\right)\right), \tag{27}$$

$$Z_i^{(m+1)} = z_i^{(m)} - \mathrm{INV}_2\left(1/u(z_i^{(m)}) - \sum_{\substack{j=1\\ j\ne i}}^{n} \mathrm{INV}_1\left(z_i^{(m)} - Z_j^{(m)} + \psi(z_j^{(m)}) \cdot s_k(m)\right)\right) \tag{28}$$

for $i \in \boldsymbol{I}_n$ and $m = 0, 1, 2, \ldots$, where $z_i^{(m)} = \operatorname{mid} Z_i^{(m)}$.

From Theorem 3 we notice that the R-order of convergence of the accelerated method (16) is at least six. The following theorem gives computationally verifiable initial conditions for the guaranteed convergence of the method (16).

Theorem 5. *Let $Z_1^{(0)}, \ldots, Z_n^{(0)}$ be inclusion disks of the zeros $\zeta_1, \ldots, \zeta_n$ of a given polynomial P. If $z_i^{(0)} = \operatorname{mid} Z_i^{(0)}$ and*

$$\max_{1\le i\le n} |\omega(z_i^{(0)})| < 1/3 \quad \text{and} \quad \eta^{(0)} > 3(n-1)r^{(0)}$$

hold, then the interval method (16) is convergent.

The proof of this theorem is similar to that given in a number of papers (see, e.g., [2,17,18,13–15]) and will be omitted.

4. Single step method with corrections

Continuing to develop inclusion methods with corrections we notice that further acceleration of the Newton-like methods may be attained using the already calculated disks in the current iteration (single step mode or Gauss–Seidel approach). Starting from (16) we can state the following single step inclusion method with Ostrowski's corrections:

$$\widehat{Z}_i = z_i - \left(\frac{1}{u(z_i)} - \sum_{j=1}^{i-1}\left(z_i - \widehat{Z}_j\right)^{I_c} - \sum_{j=i+1}^{n}\left(z_i - Z_j + \psi(z_j)\right)^{I_c}\right)^{I_c} \quad (i \in \boldsymbol{I}_n), \tag{29}$$

where $z_i = \operatorname{mid} Z_i$.

It is very difficult to find the R-order of convergence of this method. Apart from a very complicated mutual dependence of even $2n$ sequences of centers and radii of produced disks, the number of zeros n (= the polynomial degree since all the zeros are simple) is involved as a parameter. For this reason, we will use the denotation $O_R(\mathrm{IM}, n)$ for the R-order. We note that the determination of the bounds of the R-order for an arbitrary n requires an enormous labor and tedious work. However, we can estimate the limit bounds of the R-order taking the limit cases $n = 2$ and a very large n.

First, since the convergence rate of a considered single step method becomes almost the same as the one of the corresponding total step method when the polynomial degree is very large, according to Theorem 3 we have $O_R((29), n) \ge 6$ for very large n.

Table 1
The number of basic arithmetic operations.

Methods	$AS(n)$	$M(n)$	$D(n)$	$S(n)$
Newton-like method (9)	$25n^2 - n$	$10n^2 + 11n$	$4n^2$	n^2+n
(10)	$25n^2 + n$	$10n^2 + 11n$	$4n^2$	n^2+n
(16)	$33n^2 + 16n$	$13n^2 + 19n$	$4n^2 + 3n$	n^2+n

Consider now the single step method (29) for $n = 2$ and assume that $|\varepsilon_1^{(0)}| = |\varepsilon_2^{(0)}| = r_1^{(0)} = r_2^{(0)}$ (the "worst case" model). After an extensive but elementary calculation we derive the following estimates (omitting iteration index for simplicity)

$$|\hat{\varepsilon}_1| = \mathcal{O}\big(|\varepsilon_1|^3|\varepsilon_2|^3\big), \qquad |\hat{\varepsilon}_2| = \mathcal{O}\big(|\varepsilon_1|^3|\varepsilon_2|^6\big), \qquad \hat{r}_1 = \mathcal{O}\big(|\varepsilon_1|^3 r_2\big), \qquad \hat{r}_2 = \mathcal{O}\big(|\varepsilon_1|^3|\varepsilon_2|^3 r_2\big).$$

The corresponding R-matrix has the form

$$T_4 = \begin{bmatrix} 3 & 3 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 3 & 3 & 0 & 1 \end{bmatrix}$$

with $\rho(T_4) = (9 + \sqrt{45})/2 \approx 7.8541$, $\boldsymbol{x}_\rho = (0.8727, 1.4120, 0.4606, 1) > 0$. In regard to Theorem 2 we obtain

$$O_R(27, 2) \geq \rho(P_4) = 7.8541.$$

Therefore, the range of the R-order of convergence of the single step method (29) with Ostrowski's corrections and the centered inversion is (6, 7.8541). We note that the convergence order of the interval method (29) is increased in reference to the method (16) without any additional numerical operations.

5. Computational aspects

In this section we will discuss the computational efficiency of the Newton-like interval methods (9), (10) and (16) and show that the proposed method (16) is the most efficient. Then we give two numerical examples to demonstrate the convergence properties of the considered interval methods.

An estimation of computational efficiency of iterative root-finding methods provides their ranking which is of interest in designing a package of algorithms for the simultaneous determination of polynomial zeros, where automatic procedure selection is desired. The efficiency of an iterative methods (IM) can be measured in a satisfactory way using the *coefficient of efficiency* given by

$$E(\mathrm{IM}, n) = \frac{\log q}{d}, \tag{30}$$

where q is the R-order of convergence of the iterative method (IM) and d is the computational cost (see [19], [20, Chapter 5]). The ranking list of methods obtained by (30) mainly match well with a real CPU (central processor unit) time, see [21].

There are several features which should be taken in the evaluation of computational cost (processor time of a computer, taking possession of a storage space, the length of mantissa of the used (simple, double, multi)-precision arithmetic, the number of central processors available to the user, broadcasting procedure, etc.). For our purpose, using the same digital machine for the implementation of all tested methods, we will evaluate the computation cost d on the basis of arithmetic operations per iteration, taken with certain *weights* depending on processor time. These weights will be denoted by w_{AS}, w_M, w_D and w_S for addition + subtraction, multiplication, division and square root, respectively. Let $AS(n)$, $M(n)$, $D(n)$ and $S(n)$ be the number of additions + subtractions, multiplications, divisions and square roots in the realization of one iteration for all n zeros. Then the computational cost d can be (approximately) expressed as

$$d = d(n) = w_{AS}AS(n) + w_M M(n) + w_D D(n) + w_S S(n). \tag{31}$$

Hence, the formula (30) becomes

$$E(\mathrm{IM}, n) = \frac{\log q}{w_{AS}AS(n) + w_M M(n) + w_D D(n) + w_S S(n)}. \tag{32}$$

Let us note that the square root appears in the calculation of the modulus of a complex number, thus $|a + b\mathrm{i}| = \sqrt{a^2 + b^2}$. It is of practical use to take the weights appearing in (31) proportionally to the number of cycles of basic operations or the numbers of flops/s, see [22].

We will compare the interval methods (9), (10) and (16) assuming that the computer used to implement these methods would execute real arithmetic operations. The number of basic arithmetic operations of the methods (9), (10) and (16) is given in Table 1 as a function of the polynomial degree n. It is assumed that the centered inversion is applied, while multiplication and division are executed according to the formulas presented in [23].

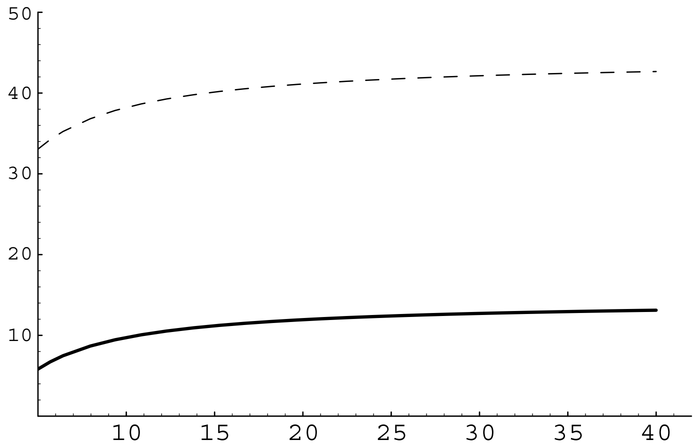

Fig. 1. Ratios of computational efficiency.

It is worth noting that, according to the analysis of efficiency given in [12, Chapter 6] for several existing computing machines, it was found that the Newton-like method (9) has the most computational efficiency. For this reason, it is unnecessary to consider other existing methods in the comparison procedure. We note that methods with corrections were not considered since they were not developed yet in that moment (1989). However, after introducing interval methods with corrections (1993), the Newton-like method with Newton's corrections (10) became the most efficient method in the class of total step simultaneous root-finding methods realized in circular complex interval arithmetic.

To compare the most efficient methods (9), (10) and (16), we used the data given in [24] for multiprecision operations. To save all significant digits and preserve the inclusion property in the case of very small disks, numerical examples were realized by the programming package Mathematica 6 relying on the GNU multiprecision package GMP developed by Granlund [25]. More details about the multiprecision arithmetic can be found, e.g., in [26].

Applying (32) we calculated the percentage ratios

$$t_{14,9}(n) = (E(14, n)/E(9, n) - 1) \cdot 100 \quad (\text{in } \%),$$
$$t_{14,10}(n) = (E(14, n)/E(10, n) - 1) \cdot 100 \quad (\text{in } \%).$$

These ratios show the (percentage) improvement of computational efficiency of the new method (16) in relation to the methods (9) and (10), and they are graphically presented in Fig. 1 as the functions of the polynomial degree n, where $t_{14,9}(n)$ is displayed by dashing line and $t_{14,10}(n)$ by full line (see Remark 5).

Remark 5. The formulas (30) and (32) for the computational efficiency are established using an empirical estimation which lead to a good coincidence with the real CPU time. The other formulas could be also applied, for example, $E = q^{1/d}$ or $E = q/d$. All of these formulas are convenient for comparison purpose and ranking the considered methods. The percentage improvement can be regarded as a relative estimation which depends on the used formula for the efficiency and the employed hardware (that is, the entries of weights in (31)).

From Fig. 1 we observe that the new interval method (16) is more efficient than the methods (9) and (10). This improvement is greater for greater n in both cases. It is considerably greater in relation to the Newton-like method (9), while it is about 10% (see Remark 5) in the case of the method (10). Other data for the weights (cycles) give slightly different results, but the average ratios of computational efficiency lead to the same conclusion. Therefore, the proposed method with Ostrowski's corrections (16) is the most efficient method for the simultaneous inclusion of polynomial zeros in the class of methods based on fixed point relations and realized in circular complex arithmetic.

Remark 6. The single step method with Ostrowski's corrections (29) possesses the fast convergence (belonging to the interval (6, 7.8541)) than the corresponding total step method (16). Therefore, it is more efficient than the method (16) and, consequently, it is the most efficient method in the considered class of simultaneous methods.

We have tested a number of polynomial equations to demonstrate the convergence behavior of the Newton-like methods (9), (10) and (16). To illustrate the convergence properties, we selected the following two examples.

Example 1. We have applied the interval methods (9), (10) and (16) for the simultaneous inclusion of zeros of the polynomial

$$P_9(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300,$$

Table 2
Inclusion disks obtained by the methods (9), (10) and (16).

Methods	$\max r_i^{(1)}$	$\max r_i^{(2)}$	$\max r_i^{(3)}$	$\max r_i^{(4)}$
Newton-like method (9)	1.1(−1)	5.70(−5)	6.10(−16)	1.50(−50)
(10)	1.1(−1)	4.57(−5)	2.16(−19)	3.01(−76)
(16)	1.1(−1)	6.40(−6)	1.70(−31)	6.10(−189)

Table 3
The values of the polynomial in the neighborhood of the zeros.

$z \in$	[−3.1, −2.9]	[−1.1, −0.9]	[0.9, 1.1]	[2.9, 3.1]	[3.9, 4.1]
$P \in$	[−5.8(13), 2.0(13)]	[−7.3(9), 1.2(10)]	[−8.6(8), 9.1(8)]	[−2.5(10), 1.5(10)]	[−5.3(11), 1.56(11)]

Table 4
Inclusion disks obtained by the methods (9), (10) and (16).

Methods	$\max r_i^{(1)}$	$\max r_i^{(2)}$	$\max r_i^{(3)}$	$\max r_i^{(4)}$
Newton-like method (9)	1.70(−1)	1.07(−4)	1.18(−15)	8.99(−50)
(10)	2.40(−1)	8.79(−4)	1.01(−14)	8.07(−57)
(16)	1.40(−1)	6.30(−4)	1.65(−18)	1.60(−105)

Table 5
Inclusion disks obtained by the methods (27) and (28) with the switch function $s_1(m)$.

Methods	$\max r_i^{(1)}$	$\max r_i^{(2)}$	$\max r_i^{(3)}$	$\max r_i^{(4)}$
(27)	1.70(−1)	1.075(−4)	3.70(−18)	1.22(−72)
(28)	1.70(−1)	1.074(−4)	4.65(−23)	1.45(−134)

starting with the initial disks $Z_i^{(0)} = \{z_i^{(0)}; 0.3\}$ that contain the exact zeros $-3, \pm 1, \pm 2\mathrm{i}, \pm 2 \pm \mathrm{i}$. The maximal radii of the inclusion disks produced in the first four iterative steps are given in Table 2, where the denotation $A(-h)$ means $A \times 10^{-h}$.

Example 2. We note above that the methods with corrections could have not extra fast convergence in the first iterations if the midpoints of inclusion disks are not close enough to the sought zeros. In such cases the corrections (say, $u(z_j)$ or $\psi(z_j)$) evaluated at the midpoints $z_j = \operatorname{mid} Z_j$ are not sufficiently small in magnitude and the effect of acceleration forced by very fast convergence of the midpoints is (partially or totally) lost. The same situation can also appear when the tested polynomial has large amplitude, as shown in the following example where we considered the algebraic polynomial of the 25th degree

$$\begin{aligned} P_{25}(z) = {} & (z-4)(z^4-1)(z^4-81)(z^2-8z+17)(z^2-6z+13)(z^2-4z+5)(z^2-2z+5) \\ & \times (z^2-4z+13)(z^2+2z+5)(z^2+4z+5)(z^2+4z+13). \end{aligned}$$

The ranges of values of the polynomial P_{25}, corresponding to the close neighborhoods of the real zeros $-3, -1, 1, 3, 4$, are given in Table 3, where $A(h)$ means $A \times 10^h$. From this table we observe that $P_{25}(z_i)$ takes very large values in magnitude so that the corrections $u(z_j)$ or $\psi(z_j)$ are also reasonable large in magnitude.

To find inclusion disks of zeros of the polynomial P_{25}, we implemented two type of algorithms; the first ran according to the iterative formulas (9), (10) and (16) in all four iterations and the results are presented in Table 4. We note that the methods (10) and (16) with corrections show considerable improvement only in the fourth iteration. In general, according to many numerical examples, it is preferable to apply in practice the combined method with switch function always when the polynomial degree is relatively high.

The second type of algorithms starts in the first iteration with the Newton-like method (9) and then continues to apply iterative formulas (10) and (16) with corrections. In other words, we dealt with the switch function $s_1(m)$, see the iterative formulas (27) and (28). The obtained maximal radii of the produced inclusion disks are given in Table 5. We face considerable improvement in the third and fourth iterations.

Acknowledgement

This work was supported by the Serbian Ministry of Science under grant 144024.

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